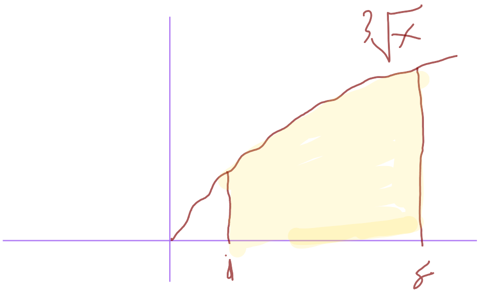
Chapter 6: Applications of Integration

Section 6.1: Areas Between Curves

Objective: In this lesson, you learn

☐ How to establish the area of a region between two curves as the limit of a Riemann sum.

Problem: Find the area bounded by $y = \sqrt[3]{x}$ and $y = \dfrac{1}{x}$, and between the vertical lines $x = 1$ and $x = 8$?



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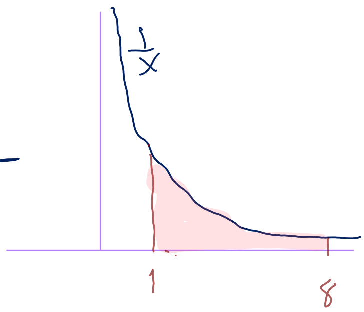
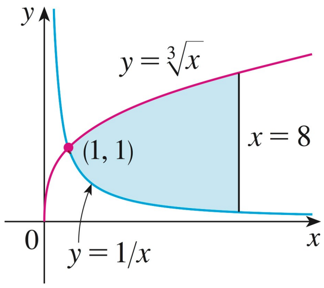




$$\int_1^8 \underbrace{\sqrt[3]{x}}_{= x^{1/3}}\, dx - \int_1^8 \frac{1}{x}\, dx = \int_1^8 \left(\sqrt[3]{x} - \frac{1}{x}\right) dx$$

$$= \frac{3}{4} x^{4/3} \Big|_1^8 - \ln|x| \Big|_1^8$$

$$\int f(x) \pm g(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

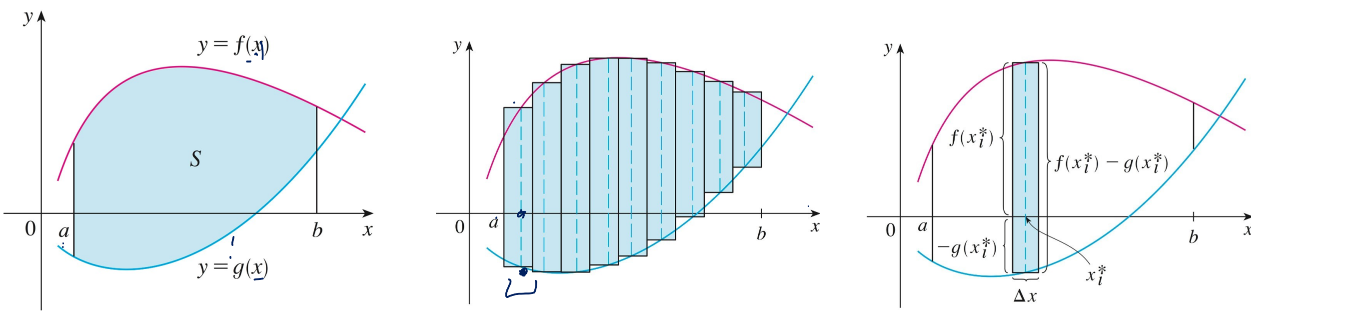
$$= \frac{3}{4}\left(8^{4/3} - 1^{4/3}\right) - \left(\ln|8| - \cancel{\ln|1|}^{\,0}\right)$$

$$= \boxed{11 - \ln 8}$$

I. Area Between Curves

Consider the region S that lies between two curves $y = f(x)$ and $y = g(x)$, and between the vertical lines $x = a$ and $x = b$, where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.

Divide S into n strips of equal width then approximate the i^{th} strip by a rectangle with base Δx and height $f(x_i^*) - g(x_i^*)$.



The Riemann sum

$$\sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \Delta x$$

is then an approximation to the area of S.

The approximation becomes better as $n \to \infty$, so define the area of the region S as

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \Delta x.$$

The limit above is the definite integral of $f - g$, we state this result as follows:

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

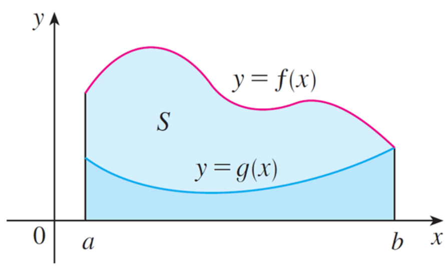
$$A = \int_a^b \left[f(x) - g(x) \right] dx.$$

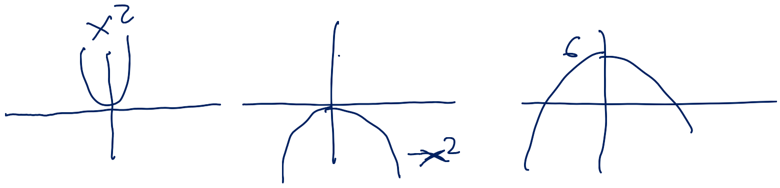
Remark

- In the case when $g(x) = 0$, S is the region under the graph of f.
- In the case when both f and g are positive,

$$A = [\text{ area under } \; y = f(x)] - [\text{ area under } \; y = g(x)]$$

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx = \int_a^b \left[f(x) - g(x) \right] dx.$$



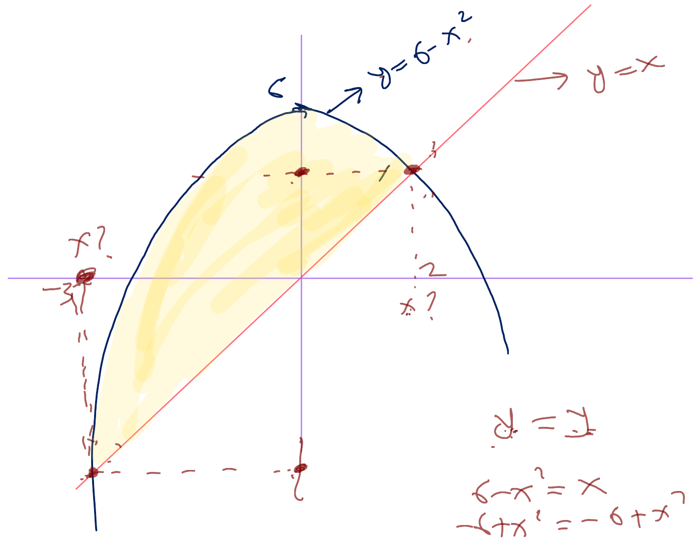


Example 1: Find the area of the region bounded by $y = x$ and parabola $y = 6 - x^2$?

$$\int_{-3}^{2} y_{\text{top}} - y_{\text{bottom}} \, dx$$

$$= \boxed{\int_{-3}^{2} (6-x^2) - (x) \, dx}$$

$$= \int_{-3}^{2} 6 - x^2 - x \, dx$$



$y = y$

$6 - x^2 = x$

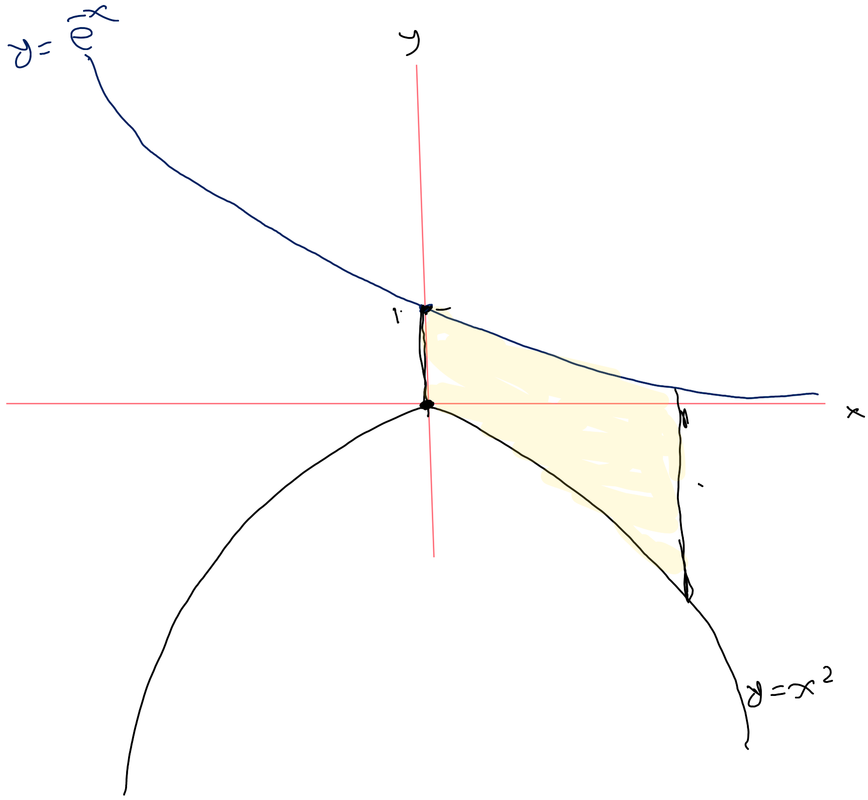
$-6 + x^2 = -6 + x^2$

$x^2 + x - 6 = 0$

$(x-2)(x+3) = 0$

$x = 2$ or $x = -3$

Example 2: Find the area of the region bounded by $y = e^{-x}$ and parabola $y = -x^2$ and between the vertical lines $x = 0$ and $x = 1$? ?



$$\int_0^1 y_{top} - y_{bottom}\,dx$$

$$\int_0^1 e^{-x} - (-x^2)\,dx$$

$$= \int_0^1 e^{-x} + x^2\,dx$$

$$= -e^{-x} + \frac{1}{3}x^3 \Big|_0^1$$

$$= \left(-e^{-1} + \frac{1}{3}\right) - \left(-e^{-0} + \cancel{\frac{1}{3}(0)}\right)$$

$$= -e^{-1} + \frac{1}{3} + 1$$

$$\int e^{-x}\,dx$$

$u = -x \rightarrow du = -dx$

$-du = dx$

$$-\int e^{u}\,du$$

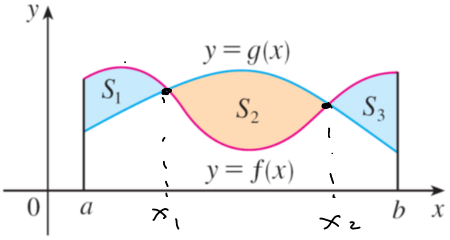
$$= -e^{u} + C$$

$$= -e^{-x} + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

In general, if $f(x) \geq g(x)$ for some values and $f(x) \leq g(x)$ for other values, then split the region S into several regions $S_1, S_2, \cdots$ with areas $A_1, A_2, \cdots$.

$$\int_a^{x_1} f(x)-g(x)\,dx + \int_{x_1}^{x_2} g(x)-f(x)\,dx + \int_{x_2}^{b} f(x)-g(x)\,dx$$



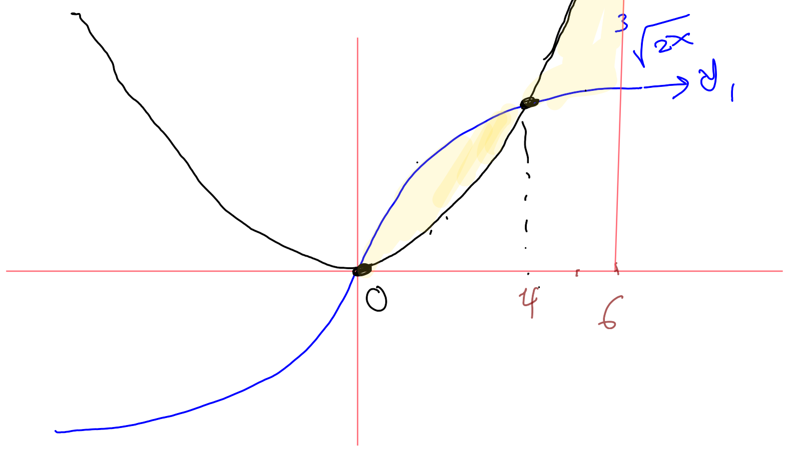
Define the area of the region S to be $A = A_1 + A_2 + \cdots$. Then, since

$$|f(x) - g(x)| = \begin{cases} f(x) - g(x), & \text{when } f(x) \geq g(x) \\ g(x) - f(x), & \text{when } g(x) \geq f(x) \end{cases}$$

we have the following expression for A: The area between the curves $y = f(x)$ and $y = g(x)$, and between the lines $x = a$ and $x = b$ is

$$A = \int_a^b |f(x) - g(x)|\,dx.$$

Example 3: Find the area of the region bounded by $y_1 = \sqrt[3]{2x}$ and parabola $y_2 = \frac{1}{8}x^2$, $0 \leq x \leq 6$?



$$\int_0^6 |y_1 - y_2|\,dx$$

$$= \int_0^4 \overset{\text{top}}{y_1} - \overset{\text{bottom}}{y_2}\,dx + \int_4^6 \overset{\text{top}}{y_2} - \overset{\text{bottom}}{y_1}\,dx$$

$$= \int_0^4 \sqrt[3]{2x} - \frac{1}{8}x^2\,dx + \int_4^6 \frac{1}{8}x^2 - \sqrt[3]{2x}\,dx$$

$y = y$

$(\sqrt[3]{2x})^3 = \left(\frac{1}{8}x^2\right)^3$

$2x = \frac{1}{8^3}x^6$

$\left(\frac{1}{8^3}x^6 - 2x\right) = 0$

$x\left(\frac{1}{8^3}x^5 - 2\right) = 0$

$\boxed{x=0}$ or $\frac{1}{8^3}x^5 - 2 = 0$

$\frac{1}{8^3}x^5 = 2$

$x^5 = 2 \cdot 8^3$

$x^5 = 2(2^3)^3$

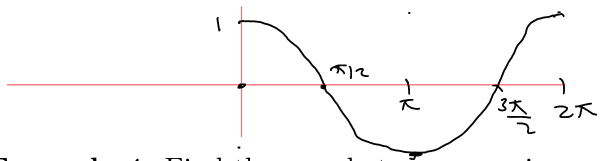
$(x^5)^{1/5} = 2 \cdot 2^9 = (2^{10})^{1/5}$

$x = 2^{10 \cdot \frac{1}{5}} = 2^2 = 4$

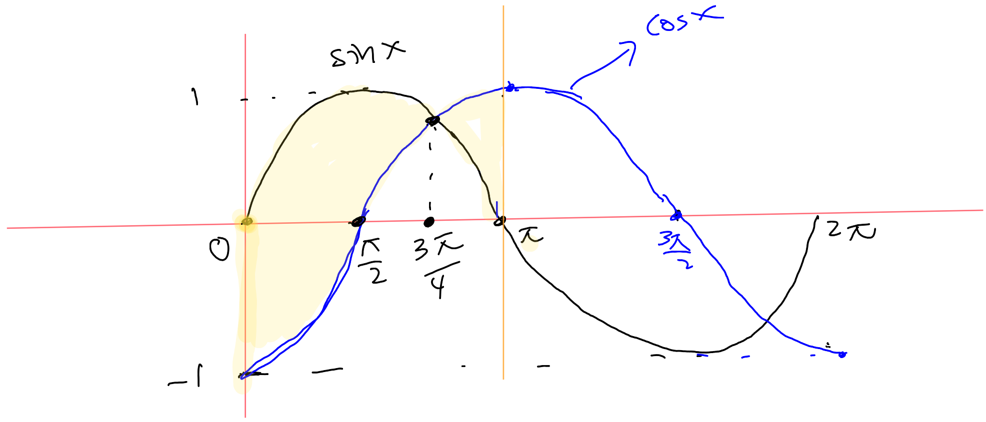
$u = 2x \rightarrow du = 2\,dx \rightarrow \frac{du}{2} = dx$

$$\int \sqrt[3]{2x}\,dx = \int (2x)^{1/3}\,dx = \frac{1}{2}\int u^{1/3}\,du$$

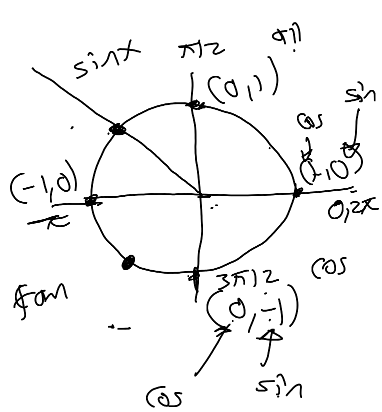
$$= 2^{1/3}\int x^{1/3}\,dx \qquad (ab)^n = a^n b^n$$



Example 4: Find the area between $y_1 = \sin x$ and $y_2 = -\cos x$, $x = 0$ and $x = \pi$?



$\sin x = -\cos x$



$$\int_0^{\pi} |y_1 - y_2|\,dx =$$

$$= \int_0^{\frac{3\pi}{4}} \sin x - (-\cos x)\,dx + \int_{\frac{3\pi}{4}}^{\pi} -\cos x - \sin x\,dx$$

$$\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$= \int_0^{3\pi/4} \sin x + \cos x\,dx - \int_{\frac{3\pi}{4}}^{\pi} \cos x + \sin x\,dx$$

$$-\cos x + \sin x \Big|_0^{\frac{3\pi}{4}} - \Big[\sin x - \cos x\Big]_{3\pi/4}^{\pi}$$

$$= \Big(-\overset{-\frac{\sqrt{2}}{2}}{\cos\frac{3\pi}{4}} + \overset{\frac{\sqrt{2}}{2}}{\sin\frac{3\pi}{4}}\Big) - \Big(\overset{1}{\cancel{\cos 0}} + \overset{0}{\cancel{\sin 0}}\Big) - \Big[\Big(\overset{0}{\cancel{\sin\pi}} - \overset{-1}{\cancel{\cos\pi}}\Big) - \Big(\overset{\frac{\sqrt{2}}{2}}{\sin\frac{3\pi}{4}} - \overset{-\frac{\sqrt{2}}{2}}{\cos\frac{3\pi}{4}}\Big)\Big]$$

$$(\sqrt{2} - 1) - ((-1) - (\sqrt{2}))$$

$$\sqrt{2} - \cancel{1} + \cancel{1} + \sqrt{2} = 2\sqrt{2}$$

$\sqrt{9} = 3$

$x^2 = 9 \Rightarrow x = \pm\sqrt{9}$

Example 5: Find the area of the region bounded by the line $y = x/2$ and parabola $y^2 = 8 - x$?

$x = 2y$ $\qquad$ $x = 8 - y^2$

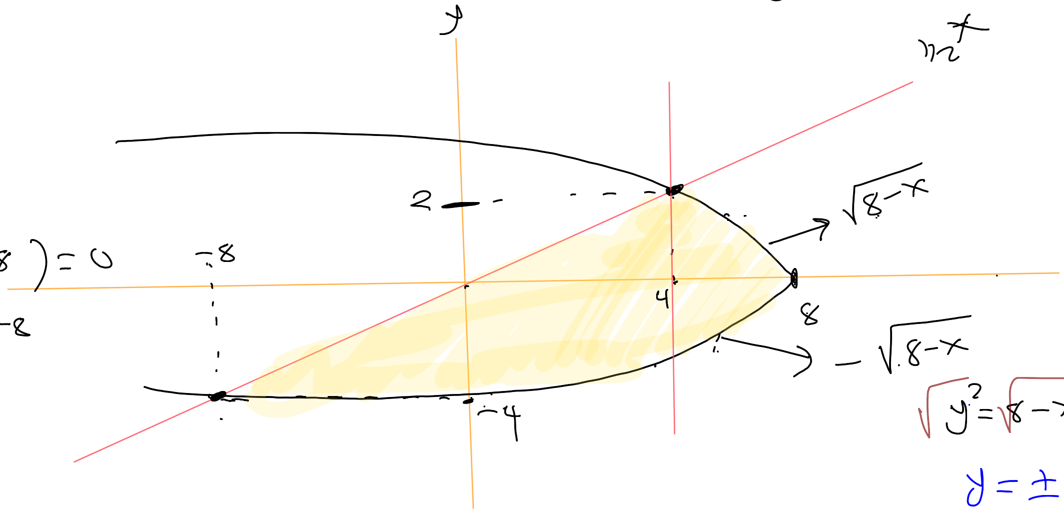
$\frac{1}{2}x = \sqrt{8-x}$

$\frac{1}{4}x^2 = 8 - x$

$x^2 + 4x - 32 = 0$

$(x-4)(x+8) = 0$

$x = 4$ or $x = -8$



$\sqrt{y^2} = \sqrt{8-x}$

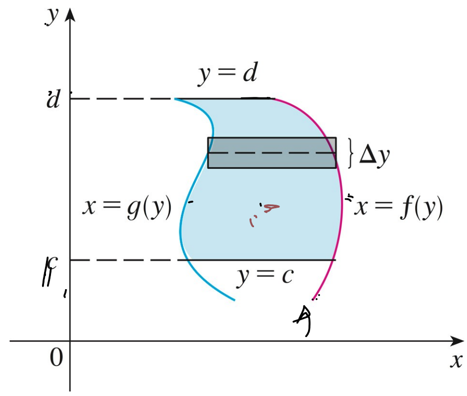
$y = \pm\sqrt{8-x}$

$$\int_{-8}^{4} \frac{1}{2}x - \left(-\sqrt{8-x}\right) dx + \int_{4}^{8} \left(\sqrt{8-x}\right) - \left(-\sqrt{8-x}\right) dx$$

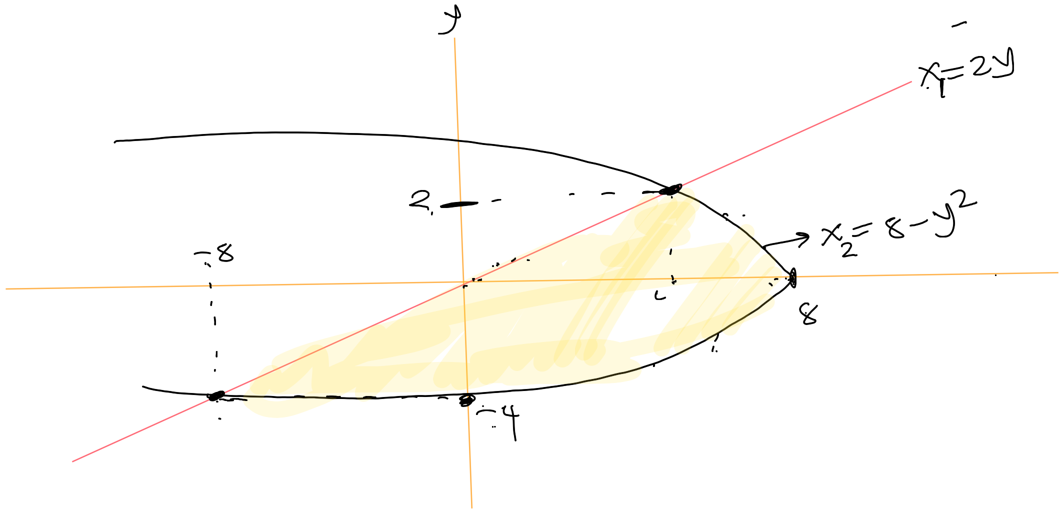
Integrating along y-axis

Suppose the area A is bounded by the curve $x = f(y)$, $x = g(y)$, and the lines $y = c$, $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for all $y \in [c, d]$.
Then the area is given by

$$A = \int_c^d [f(y) - g(y)]dy$$



Example 6: Find the area of the region bounded by the line $y = x/2$ and parabola $y^2 = 8 - x$?



$$\int_{-4}^{2} x_2 - x_1 \, dy = \int_{-4}^{2} (8-y^2) - (2y) \, dy$$

$$= 8y - \frac{y^3}{3} - y^2 \Big|_{-4}^{2}$$